Boundary Conditions

The first order approximation for the surface boundary conditions on pressure and temperature are

$$P(m = \mathcal{M}_T) = 0 \tag{4.1.3}$$

$$T(m = \mathcal{M}_T) = 0 \tag{4.1.4}$$

Compared to the values for temperature and pressure within the star, small values such as $P = T = 0$ for the surface conditions are valid. However, we can improve upon these boundary conditions by using relations between surface temperature, pressure, luminosity, and radius that come from the analysis of stellar atmospheres.

To estimate the pressure and temperature in the atmosphere of a star with luminosity, $\mathcal{L}_T$, radius, R, and mass $\mathcal{M}_T$, recall that for radiative diffusion

$$F_\nu = \frac{\mathcal{L}_\nu}{4\pi r^2} = -D\,\nabla n = -\frac{c}{3\kappa_\nu \rho}\nabla u_\nu \tag{9.1}$$

where u is the energy density, and we have explicitly labeled the frequency dependence of u and κ, and $\mathcal{L}$. Since the energy density is related to the radiation pressure by

$$P_{\rm rad} = \frac{aT^4}{3} = \frac{u}{3} \tag{7.1.3}$$

(9.1) is equivalent to

$$\frac{\mathcal{L}_\nu}{4\pi r^2} = -\frac{c}{3\kappa_\nu \rho}\nabla(3P_{\rm rad}) = \frac{c}{\kappa_\nu \rho}\frac{dP_{\rm rad}}{dr} \tag{9.2}$$

To simplify this equation, we first get rid of the frequency dependence by finding a mean value for the opacity. If we write (9.2) without the ν dependence,

$$\frac{\mathcal{L}}{4\pi r^2} = \frac{c}{\bar{\kappa}\rho}\frac{dP_{\rm rad}}{dr} \tag{9.3}$$

then it is clear from a comparison of the two equations that

$$\bar{\kappa} = \frac{1}{\mathcal{L}}\int_0^\infty \kappa_\nu \mathcal{L}_\nu d\nu \tag{9.4}$$

Note that this is not a Rosseland mean opacity (but often a Rosseland mean is used).

If we now substitute $d\tau = \bar{\kappa}\,\rho\,dr$ in (9.3), the equation becomes

$$\frac{dP_{\rm rad}}{d\tau} = \frac{\mathcal{L}}{4\pi r^2 c} \tag{9.5}$$

Compared with the stellar radius, the atmosphere is very thin, with $r \approx R$. Thus we can trivially integrate (9.5) from the stellar "surface" ($\tau = 0$) to a given optical depth, τ_p, by

$$P_{\rm rad}(\tau = \tau_p) - P_{\rm rad}(\tau = 0) = \int_0^{\tau_p} \frac{\mathcal{L}}{4\pi R^2 c} d\tau = \frac{\mathcal{L}}{4\pi R^2 c}\tau_p \tag{9.6}$$

Note here that $P_{\rm rad}(\tau = 0)$ will not be $u/3$, since that implies an isotropic radiation field. (Recall, that in deriving the diffusion equation, we assumed that there were just as many particles diffusing inward across a boundary, as there were particles diffusing outward.) If we make the simple assumption that photons only diffuse outward at the stellar surface, then $P_{\rm rad}(\tau = 0)$ will be 1/2 the isotropic value, *i.e.*, $P_{\rm rad}(\tau = 0) = u/6$. In thermodynamic equilibrium, $u_\nu = 4\pi B(T)/c$, which means

$$P_{\rm rad}(\tau_p) = \frac{\mathcal{L}}{4\pi R^2 c}\tau_p + \frac{1}{6}\frac{4\pi B(T)}{c} = \frac{\mathcal{L}}{4\pi R^2 c}\tau_p + \frac{2\pi}{3c}B(T)$$

If we now *define* the effective temperature of an object through the equation $\mathcal{L}_T = 4\pi R^2 \sigma T_{\text{eff}}^4$, then $\pi B(T) = \sigma T_{\text{eff}}^4$, and

$$P_{\text{rad}}(\tau_p) = \frac{\sigma}{c} T_{\text{eff}}^4 \, \tau_p + \frac{2}{3c} \sigma T_{\text{eff}}^4 = \frac{\sigma}{c} \left(\tau_p + \frac{2}{3} \right) T_{\text{eff}}^4 \tag{9.7}$$

This approximates how the radiation pressure of a star changes with optical depth. Similarly, if we relate the radiation pressure to the temperature by $P_{\text{rad}} = aT^4/3$ (and remember that $a = 4\sigma/c$), then

$$T^4(\tau_p) = \frac{3}{4} T_{\text{eff}}^4 \left(\tau_p + \frac{2}{3} \right) \tag{9.8}$$

Note that (9.8) defines what the optical depth is at the location of the photosphere, *i.e.*, at $T = T_{\text{eff}}$, $\tau_p = 2/3$.

Since we now know where the stellar photosphere is in terms of τ_p, we can use this to compute the pressure at the photosphere. In hydrostatic equilibrium,

$$dP = -\frac{G\mathcal{M}_T}{R^2} \rho \, dr = -\frac{G\mathcal{M}_T}{\bar{\kappa} R^2} d\tau \tag{9.9}$$

If we integrate this equation from the surface ($\tau = 0$) to the photosphere ($\tau = 2/3$), and make the (somewhat poor) assumption that the opacity doesn't change with optical depth, then the photospheric pressure, P_p, is

$$P_p = \frac{2G\mathcal{M}_T}{3\bar{\kappa} R^2} + P(\tau = 0) \tag{9.10}$$

Now let's assume that at $\tau = 0$ the gas pressure is negligible compared to the radiation pressure. With this assumption, we again have

$$P(\tau = 0) = \frac{2\pi}{3c} B(T) = \frac{2}{3c} \sigma T_{\text{eff}}^4$$

which yields

$$P_p = \frac{2G\mathcal{M}_T}{3\bar{\kappa}R^2} + \frac{2}{3c}\frac{\mathcal{L}_T}{4\pi R^2} = \frac{2}{3}\frac{G\mathcal{M}_T}{\bar{\kappa}R^2}\left(1 + \frac{\bar{\kappa}\mathcal{L}_T}{4\pi c\, G\mathcal{M}_T}\right) \quad (9.11)$$

The second term in (9.11) is the ratio of the stellar luminosity to the Eddington luminosity. Under most circumstances, $\mathcal{L}_T \ll 4\pi c\, G\mathcal{M}_T/\bar{\kappa}$, so the latter term can be neglected. This leaves

$$P_p = \frac{2}{3}\frac{G\mathcal{M}_T}{\bar{\kappa}R^2} \quad (9.12)$$

Matching Radiative Atmospheres to Interiors

Although the stellar photosphere boundary conditions represent an improvement over the surface boundary conditions, they still are not ideal. At the photosphere, the mean free path of a photon is long compared to the pressure, density, and temperature gradients. This violates the assumptions that went into deriving the equations of radiative energy transport. To avoid this problem, this atmospheric grid should be used to match the interior models below the photosphere.

To see how this can work, let's adopt an opacity law of the form

$$\kappa = \kappa_0 \rho^p T^q = \kappa_0' P^\alpha T^\beta \tag{9.13}$$

where (for an ideal gas)

$$\alpha = p; \qquad \beta = q - p; \qquad \kappa_0' = \kappa_0 \left(\frac{\mu\, m_a}{k} \right)^\alpha$$

If the envelope is radiative, *i.e.*,

$$\nabla_{\rm rad} = \frac{d \ln T}{d \ln P} = \frac{3\kappa}{16\pi ac\, G} \frac{P}{T^4} \frac{\mathcal{L}_T}{\mathcal{M}} \tag{3.1.6}$$

then

$$\nabla_{\rm rad} = \frac{P}{T} \frac{dT}{dP} = \frac{3\kappa_0' \mathcal{L}_T}{16\pi ac\, G\mathcal{M}} P^{1+\alpha}\, T^{\beta-4} = C_1\, P^{1+\alpha}\, T^{\beta-4} \tag{9.14}$$

which implies

$$\int_{T_p}^{T} T^{3-\beta} dT = C_1 \int_{P_p}^{P} P^\alpha dP$$

or

$$T^{4-\beta}\left\{1-\left(\frac{T_p}{T}\right)^{4-\beta}\right\}=\frac{4-\beta}{1+\alpha}\,C_1\,P^{1+\alpha}\left\{1-\left(\frac{P_p}{P}\right)^{1+\alpha}\right\} \tag{9.15}$$

This expression explains why one can use surface boundary conditions ($P=0$, $T=0$) in a stellar model. If $\beta<4$, and $\alpha>-1$, (as is true for most opacities), then as one goes deeper into the interior of the star,

$$T^{4-\beta}\longrightarrow\frac{4-\beta}{1+\alpha}\,C_1\,P^{1+\alpha} \tag{9.16}$$

making P and T independent of the surface conditions. Moreover, a comparison of (9.14) and (9.16) yields

$$\nabla_{\rm rad}=\frac{1+\alpha}{4-\beta} \tag{9.17}$$

which allows us to relate the temperature to the star's radius. Recall that

$$\frac{dT}{dr}=-\frac{G\mathcal{M}}{r^2}\frac{\rho T}{P}\nabla_{\rm rad} \tag{2.4.3}$$

If we assume that the amount of mass contained in surface layers is negligible, and use the ideal gas law, then

$$\int_{T_p}^{T}dT=-\left(\frac{\mu m_H}{k}\right)\left(\frac{1+\alpha}{4-\beta}\right)\int_R^r\frac{G\mathcal{M}_T}{r^2}dr$$

or

$$T(r)-T_p=\left(\frac{\mu m_H}{k}\right)\left(\frac{1+\alpha}{4-\beta}\right)G\mathcal{M}_T\left(\frac{1}{r}-\frac{1}{R}\right) \tag{9.18}$$

We can thus determine boundary conditions deeper in the star, and reach a region where the stellar interior approximations hold.

Convective Atmospheres

For cool stars, H^- opacity dominates the surface layers. Since, $\beta \sim 9$ for this type of opacity, the analysis above does not apply. To see what happens, consider the behavior of temperature and pressure in a small radiative region on the surface of the star. At the surface, the radiative temperature gradient is

$$\nabla_{\rm rad} = \nabla_p = \frac{3\kappa\mathcal{L}_T}{16\pi ac\, G\mathcal{M}_T}\frac{P_p}{T_{\rm p}^4} \tag{9.19}$$

or, if we again assume $\kappa = \kappa_0'\, P^\alpha T^\beta$,

$$\nabla_p = \frac{3\kappa_0\mathcal{L}_T}{16\pi acG\mathcal{M}_T}\frac{P_p^{\alpha+1}}{T_p^{4-\beta}} = C_1\, P_p^{\alpha+1}\, T_p^{\beta-4} \tag{9.20}$$

Now, re-write (9.15) as

$$\left(T^{4-\beta} - T_p^{4-\beta}\right) = \left(\frac{4-\beta}{\alpha+1}\right) C_1\, \left\{P^{\alpha+1} - P_p^{\alpha+1}\right\} \tag{9.21}$$

divide through by $T^{4-\beta}$, and substitute in $\nabla_{\rm rad}$ and ∇_p for P and P_p

$$\left\{1 - \left(\frac{T_p}{T}\right)^{4-\beta}\right\} = \left(\frac{4-\beta}{\alpha+1}\right)\left\{\nabla_{\rm rad} - \nabla_p\left(\frac{T_p}{T}\right)^{4-\beta}\right\}$$

The expression for $\nabla_{\rm rad}$ at some depth in the star is then

$$\nabla_{\rm rad} = \nabla_p\left(\frac{T_p}{T}\right)^{4-\beta} + \left(\frac{\alpha+1}{4-\beta}\right)\left\{1 - \left(\frac{T_p}{T}\right)^{4-\beta}\right\}$$

or

$$\nabla_{\rm rad} = \frac{\alpha+1}{4-\beta} + \left(\frac{T_p}{T}\right)^{4-\beta}\left\{\nabla_p - \frac{\alpha+1}{4-\beta}\right\} \tag{9.22}$$

This can be further simplified if we evaluate ∇_p at the stellar photosphere. If we substitute

$$T_p = T_{\rm eff}; \qquad \mathcal{L}_T = 4\pi R^2 \sigma T_{\rm eff}^4; \qquad P_p = \frac{2}{3}\frac{G\mathcal{M}_T}{\kappa R^2}$$

(the latter from (9.12)), and remember that $a = 4\sigma/c$, then

$$\nabla_p = \frac{3\kappa}{16\pi ac G\mathcal{M}_T T_{\rm eff}^4}\left(4\pi R^2 \sigma T_{\rm eff}^4\right)\left(\frac{2}{3}\frac{G\mathcal{M}_T}{\kappa R^2}\right) = \frac{1}{8} \qquad (9.23)$$

Thus, at $T = T_{\rm eff}$, $\tau_p = 2/3$, and $\nabla_p = 1/8$.

Now, consider a cool star, in which H^- opacity in important. At low temperatures, $\kappa \propto \rho^{1/2}T^9$, and thus $\alpha = 1/2$ and $\beta = 17/2$. Equation (9.22) for $\nabla_{\rm rad}$ then gives

$$\nabla_{\rm rad}(r) = -\frac{1.5}{4.5} + \left(\frac{T_p}{T}\right)^{-4.5}\left(\frac{1}{8} - \frac{1.5}{-4.5}\right) = -\frac{1}{3} + \frac{11}{24}\left[\frac{T(r)}{T_{\rm eff}}\right]^{9/2}$$

The equation demonstrates that $\nabla_{\rm rad}$ will increase rapidly as the temperature increases towards the interior of the star. In fact, just below the surface, $\nabla_{\rm rad}$ will become greater than $\nabla_{\rm ad}$, Thus, the region just below the surface will be convectively unstable. This is a feature of all cool stars: if H^- is the dominant source of opacity, convection will occur. If we assume that convection will begin when $\nabla_{\rm rad} > 0.4$, then the temperature at this transition zone will be

$$\left[\frac{T}{T_{\rm eff}}\right]^{9/2} = \left(\frac{2}{5} + \frac{1}{3}\right)\left(\frac{24}{11}\right) \implies T_t = \left(\frac{8}{5}\right)^{2/9} T_{\rm eff}$$

$$\approx 1.11\, T_{\rm eff} \qquad (9.24)$$

In a similar fashion, we can estimate the pressure at the transition zone. If we take (9.21) and substitute for the constant C_1 using (9.20), then

$$T^{4-\beta} - T_p^{4-\beta} = \left(\frac{4-\beta}{\alpha+1}\right) \left(\frac{\nabla_p}{P_p^{\alpha+1}\, T_p^{\beta-4}}\right) \{P^{\alpha+1} - P_p^{\alpha+1}\}$$

or

$$\left(\frac{T}{T_p}\right)^{4-\beta} - 1 = \left(\frac{4-\beta}{\alpha+1}\right) \nabla_p \left\{\left(\frac{P}{P_p}\right)^{\alpha+1} - 1\right\} \tag{9.25}$$

From (9.23) and (9.24), $\nabla_p = 1/8$ and $T/T_p = (8/5)^{2/9}$, so this equation can be directly solved for P. When we plug in the numbers for the $\alpha = 1/2$, $\beta = 17/2$ case of H^- opacity, then the pressure at the transition zone becomes

$$\left(\frac{P_t}{P_p}\right)^{\alpha+1} = 2 \quad \Longrightarrow \quad P_t = 2^{2/3} P_p \tag{9.26}$$